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Energy loss from an electron pulse in a plasma

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Abstract. We present upper and lower bounds for the rate of energy loss by a high intensity Gaussian pulse of finite length propagating in a cold dense homogeneous plasma. For sufficiently high plasma conductivity these bounds coalesce yielding an explicit expression for the dissipated power.

1. Introduction

The purpose of this paper is to discuss the power P lost by a high intensity relativistic electron pulse propagating in a plasma and to demonstrate that, in the special case of a Gaussian pulse, upper and lower bounds can be obtained for P. If the conductivity of the plasma is sufficiently large, these bounds coincide thereby yielding an explicit result for P.

We will consider a single high intensity pulse of finite length propagating in an homogeneous cold dense plasma of infinite extent. It is well known (Cox and Bennett 1970, Hammer and Rostoker 1970, Lee and Sudan 1971, Lovelace and Sudan 1971)† that such a pulse induces a 'return current' and with it an electric field $\boldsymbol{E}$ so that the power lost is given by

$$P=\int \boldsymbol{J}\,.\,\boldsymbol{E}\,\mathrm{d}^3x \qquad (1)$$

where $\boldsymbol{J}$ denotes the beam current density.

It will be supposed that the plasma can be treated as a conducting medium with constant conductivity and that it is a good conductor so that the displacement current can be neglected in Maxwell's equations. In addition, we shall assume that the beam–plasma system has cylindrical symmetry about the z axis and omit consideration of any beam–plasma instabilities. The latter condition seems to be reasonable because the collective interaction of the beam electrons with the plasma is expected to be weak for high energy beams and small beam–plasma density ratios (Lovelace and Sudan 1971).

2. Formulation and evaluation

Under the above assumptions we find the following expression for the electric field $\boldsymbol{E}$ in

† Many more references are given by Davidson and Hui (1975).

terms of the beam current density $\boldsymbol{J}$ (using MKS units):

$$\boldsymbol{E}(\boldsymbol{x}, t) = \int \dots \int G(x - x', t - t')\left(-\frac{1}{\sigma}\frac{\partial}{\partial t'}\boldsymbol{J} + \frac{D}{\sigma}\boldsymbol{\nabla}'(\boldsymbol{\nabla}' \cdot \boldsymbol{J})\right) \mathrm{d}^3x' \, \mathrm{d}t' \tag{2}$$

where

$$G(\boldsymbol{x}, t) = \theta(t)(4\pi Dt)^{-3/2} \exp[-(\boldsymbol{x})^2/4Dt]$$

denotes the customary Green function for the diffusion equation, $\theta(t)$ being the usual Heaviside step function

$$\theta(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0. \end{cases}$$

We have defined a diffusion coefficient $D = (\mu_0\sigma)^{-1}$ and have used the relation

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = -\frac{1}{\sigma}\boldsymbol{\nabla} \cdot \boldsymbol{J},$$

which follows from Maxwell's equations.

We will take the beam current density in the form,

$$\boldsymbol{J} = \boldsymbol{e}_z J = \boldsymbol{e}_z I\theta(vt - z)\theta(z - v(t - \tau))g(\boldsymbol{r}) \tag{3}$$

where I denotes the current and we have introduced cylindrical coordinates. $\boldsymbol{e}_z$ denotes a unit vector along the positive z axis, and $g(\boldsymbol{r})$ the (two-dimensional) radial extension of the pulse. For convenience we have introduced the pulse duration τ so that the pulse length is $L = v\tau$, v being the pulse velocity which will hereafter be taken as the velocity of light c.

It will be convenient to have a separate notation for the first term of (2):

$$\boldsymbol{E}_0(\boldsymbol{r}, z, t) = \boldsymbol{e}_z E_0(\boldsymbol{r}, z, t) = \boldsymbol{e}_z \int \dots \int G(\boldsymbol{r} - \boldsymbol{r}', z - z')\left(-\frac{1}{\sigma}\frac{\partial J}{\partial t'}\right) \mathrm{d}^2r' \, \mathrm{d}z' \, \mathrm{d}t'.$$

In order to evaluate the remaining term in (2) we note from (3) that the beam current density is parallel to the z axis and easily derive the following relation:

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} = \frac{\partial J}{\partial z} = -\frac{1}{c}\frac{\partial J}{\partial t}, \tag{4}$$

so that we can write the total electric field (2) in the form,

$$\boldsymbol{E}(\boldsymbol{r}, z, t) = \boldsymbol{e}_z\left(1 + \frac{D}{c}\frac{\partial}{\partial z}\right) E_0(\boldsymbol{r}, z, t) + \boldsymbol{E}_1(\boldsymbol{r}, z, t). \tag{5}$$

In obtaining this expression we have decomposed the gradient operator in (2) into its components along the longitudinal and radial directions. The term in (5) involving the z derivative arises from the longitudinal gradient wherein we have used a familiar argument to replace the operator $\nabla_{z'}$ by $-\nabla_z$. As usual, this involves an integration by parts in which one shows that the contribution 'at infinity' is zero. This part of the total electric field can also be obtained by using (3), (4) and the following familiar property of the derivative of the Dirac delta function δ':

$$\int_{-\infty}^{\infty} \delta'(x)\phi(x)\,\mathrm{d}x = -\phi'(0). \tag{6}$$

This procedure is clearly justified in the present case because one uses (6) with ϕ a Gaussian function. The field $\boldsymbol{E}_1$ arises from the radial gradients and will not be considered because it does not contribute to (1).

We will choose our radial function in the form of a normalized Gaussian,

$$g(\boldsymbol{r}) = (4\pi\rho)^{-1}\exp\left(-\frac{(\boldsymbol{r})^2}{4\rho}\right), \qquad 0<\rho<\infty. \tag{7}$$

Physically, the choice of this function amounts to the consideration of a diffuse boundary for the pulse. In addition, we can obtain results for the limiting case of a pulse of zero width because g is an approximation to a two-dimensional Dirac delta function. It will be seen that P is divergent in the limit $\rho \to 0$.

In order to compare our results with previous approximate calculations for a uniform pulse (M Lampe 1976, private communication) it is convenient to relate the parameter ρ to the radius a of an equivalent uniform pulse. This we do by equating the root-mean-square values of the distribution (7) to the root-mean-square radius for a uniform pulse, thereby obtaining

$$\rho = a^2/8. \tag{8}$$

We now want to compute the rate of energy loss from the pulse using the radial function (7). Invoking (3) and (5), one contribution to (1) is

$$\begin{aligned} P_0 &\equiv \int E_0 J\,\mathrm{d}^2r\,\mathrm{d}z \\ &= \frac{-I^2c}{(4\pi)^{3/2}\sigma D^{1/2}}\int_{c(t-\tau)}^{ct}\mathrm{d}z\int_0^{\infty}\mathrm{d}s\,s^{-1/2}\left(\frac{a^2}{4}+Ds\right)^{-1}\exp\left(-\frac{c^2s}{4D}\right) \\ &\quad\times\left[\exp\left(-\frac{(z-ct)^2}{4Ds}-\frac{c(z-ct)}{2D}\right)\right. \\ &\quad\left.-\exp\left(-\frac{(z-c(t-\tau))^2}{4Ds}-\frac{c(z-c(t-\tau))}{2D}\right)\right]. \end{aligned} \tag{9}$$

The remaining contribution to P can be obtained by first computing $\partial E_0/\partial z$:

$$\begin{aligned} P_1 &\equiv \frac{D}{c}\int \frac{\partial E_0}{\partial z} J\,\mathrm{d}^2r\,\mathrm{d}z \\ &= \frac{I^2}{2\sigma(4\pi)^{3/2}D^{1/2}}\int_{c(t-\tau)}^{ct}\mathrm{d}z\int_0^{\infty}\mathrm{d}s\,s^{-3/2}\left(\frac{a^2}{4}+Ds\right)^{-1}\exp\left(-\frac{c^2s}{4D}\right) \\ &\quad\times\left[(z-ct+cs)\exp\left(-\frac{(z-ct)^2}{4Ds}-\frac{c(z-ct)}{2D}\right)\right. \\ &\quad\left.-(z-c(t-\tau)+cs)\exp\left(-\frac{(z-c(t-\tau))^2}{4Ds}-\frac{c(z-c(t-\tau))}{2D}\right)\right]. \end{aligned} \tag{10}$$

The integrations over s in (9) and (10) can be performed by the method summarized in the appendix so that we find:

$$-\frac{P}{I^2}=Q+\frac{L}{2\pi\sigma a^2}+(240\pi^2\sigma^2a^2)^{-1}(1-e^{-120\pi L\sigma}) \tag{11}$$

where

$$Q=(2\sigma a\pi^{1/2})^{-1}\int_{\alpha}^{\alpha+\beta}[2\alpha-y(1+e^{-4\alpha(y-\alpha)})]\,e^{y^2}\operatorname{erfc}(y)\,dy \tag{12}$$

in terms of the complementary error function

$$\operatorname{erfc}(x)=2\pi^{-1/2}\int_x^{\infty}e^{-t^2}\,dt,$$

and we have introduced the parameters

$$\alpha=ca/4D=30\pi\sigma a \qquad \text{and} \qquad \beta=L/a.$$

Since P denotes the power dissipated by the pulse, it will be negative and we are in a position to obtain upper and lower bounds for the positive quantity $-P/I^2$. This we do by using the estimates (Abramowitz and Stegun 1964):

$$[x+(x^2+2)^{1/2}]^{-1}\leqslant e^{x^2}\int_x^{\infty}e^{-t^2}\,dt\leqslant[x+(x^2+4\pi^{-1})^{1/2}]^{-1}, \qquad x\geqslant 0. \tag{13}$$

Thus, we find the following upper and lower bounds for Q:

$$\begin{aligned}Q_{\max}=30\bigg[&\ln\left(\frac{\alpha+\beta+[(\alpha+\beta)^2+4\pi^{-1}]^{1/2}}{\alpha+(\alpha^2+4\pi^{-1})^{1/2}}\right)\\&+\tfrac{1}{4}\pi\{(\alpha+\beta)[(\alpha+\beta)^2+4\pi^{-1}]^{1/2}-\alpha(\alpha^2+4\pi^{-1})^{1/2}+\alpha^2-(\alpha+\beta)^2\}\\&+(6\alpha)^{-1}(1+e^{-4\alpha\beta})\{(\alpha+\beta)^3-\alpha^3+(\alpha^2+2)^{3/2}-[(\alpha+\beta)^2+2]^{3/2}\}\bigg]\end{aligned} \tag{14}$$

and

$$\begin{aligned}Q_{\min}=30\bigg[&\ln\left(\frac{\alpha+\beta+(\alpha+\beta)^2+2)^{1/2}}{\alpha+(\alpha^2+2)^{1/2}}\right)\\&+\tfrac{1}{2}\{(\alpha+\beta)[(\alpha+\beta)^2+2]^{1/2}-\alpha(\alpha^2+2)^{1/2}+\alpha^2-(\alpha+\beta)^2\}\\&+\pi(6\alpha)^{-1}\{(\alpha+\beta)^3-\alpha^3+(\alpha^2+4\pi^{-1})^{3/2}-[(\alpha+\beta)^2+4\pi^{-1}]^{3/2}\}\bigg].\end{aligned} \tag{15}$$

In deriving (14) and (15) from (12) we have used (13) and have eliminated the exponential factor in the integrand of (12) by means of the inequalities $e^{-4\alpha\beta}\leqslant e^{-4\alpha(y-\alpha)}\leqslant 1$.

It follows from (11) and (15) that P diverges in the limit $a\to 0$.

Returning to the case $a\neq 0$, the above expressions for $Q_{\max}$ and $Q_{\min}$ simplify considerably in the approximation $\alpha^2\gg 2$. This condition amounts to the requirement that the plasma be a good conductor since it is equivalent to

$$\sigma a\gg 2^{1/2}/30\pi. \tag{16}$$

In this connection we recall that a large conductivity has already been assumed in omitting the displacement current from Maxwell's equations.

In the approximation (16) $Q_{max} = Q_{min}$ with only the logarithmic terms surviving and we obtain

$$-\frac{P}{I^2} = 30\left[\ln\left(1+\frac{L}{30\pi\sigma a^2}\right)+\frac{1}{2}\frac{L}{30\pi\sigma a^2}\right]. \tag{17}$$

We have also simplified the non-logarithmic term since inequality (16) implies $L\sigma \gg 2^{1/2}L/30\pi a$ and we always assume a beam radius much smaller than the pulse length.

In the approximation (16) we can use the well known asymptotic expansion for the complementary error function (Abramowitz and Stegun 1964) in (12). One easily verifies that substitution of the first term of this expansion in that equation leads to a value for $-P/I^2$ which is precisely the logarithmic term in (17). The second term in (17) represents (approximately) a summation of the remaining terms in the asymptotic expansion. However, one does not easily obtain this result from the asymptotic expansion approach.

3. Concluding remarks

We have obtained an expression for the rate of energy loss from a high intensity Gaussian pulse of finite length propagating in a cold dense homogeneous plasma of infinite extent.

Subject to the approximation (16), our result (17) consists of a logarithmic term plus a term linear in the pulse length L. By virtue of the relation (8) between our parameter ρ and the radius a of an equivalent uniform pulse (in the root-mean-square sense discussed in § 2) the logarithmic term agrees with approximate calculations for a uniform pulse performed independently by M Lampe and E P Lee (1976, private communication). Insofar as we are aware, the term linear in L has not previously been obtained. Under the assumptions (16) and $a \ll L$ this term can be comparable with the logarithmic term. In fact, this term is always at least 1/2 of the logarithmic term as is easily shown by means of the inequality

$$\ln(1+x) \leqslant x, \qquad x \geqslant 0.$$

In the event that (16) is not satisfied but the displacement current can still be neglected in Maxwell's equations, one can resort to (11) and the bounds (14) and (15).

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Appendix

Consider the integral,

$$G(a,b;s) = \int_0^\infty t^{-1/2}(t+b)^{-1}\, e^{-at^{-1}}\, e^{-st}\, dt \tag{A.1}$$

for $s \geqslant 0$, $a > 0$, and $b > 0$. This is needed to evaluate P_0 and P_1, given respectively by (9) and (10). A different integral is also needed in the evaluation of P_1, but it is readily obtained from (A.1) by differentiation with respect to the parameter a.

The integral defined in (A.1) does not appear in the standard tables so that a short discussion of our method of evaluation of it may be of interest.

Consider (A.1) as a function of s, treating a and b as parameters. One finds the equation

$$-\frac{\mathrm{d}G}{\mathrm{d}s}+bG=\int_0^\infty t^{-1/2}\,\mathrm{e}^{-at^{-1}}\,\mathrm{e}^{-st}\,\mathrm{d}t=\left(\frac{\pi}{s}\right)^{1/2}\mathrm{e}^{-2(as)^{1/2}} \tag{A.2}$$

with the boundary condition,

$$G(s=0)=\int_0^\infty t^{-1/2}(t+b)^{-1}\,\mathrm{e}^{-at^{-1}}\,\mathrm{d}t=\pi b^{-1/2}\,\mathrm{e}^{a/b}\,\mathrm{erfc}((a/b)^{1/2}). \tag{A.3}$$

It is then a straightforward matter to integrate (A.2) using (A.3) and standard techniques (integrating factor method). One finds

$$G(a,b;s)=\pi b^{-1/2}\,\mathrm{e}^{a/b}\,\mathrm{e}^{bs}\,\mathrm{erfc}((bs)^{1/2}+(a/b)^{1/2}).$$

References

Abramowitz M and Stegun I A (eds) 1964 *Handbook of Mathematical Functions: NBS Appl. Math. Ser. No. 55* (Washington, DC: US Government Printing Office) p 298
Cox J L Jr and Bennett W H 1970 *Phys. Fluids* **13** 182–92
Davidson R C and Hui B H 1975 *Ann. Phys., NY* **94** 209–42
Hammer D A and Rostoker N 1970 *Phys. Fluids* **13** 1831–50
Lee R and Sudan R N 1971 *Phys. Fluids* **14** 1213–25
Lovelace R V and Sudan R N 1971 *Phys. Rev. Lett.* **27** 1256–9